AD-A159 163 NON-UNIFORM BOUNDS OF NORMAL APPROXIMATION FOR FINITE-POPULATION U-STATIS..(U) PITTSBURGH UNIV PA CENTER FOR MULTIVARIATE ANALYSIS M BAIQI ET AL. JUL 85 1/1

UNCLASSIFIED TR-85-26 AFOSR-TR-85-0696 F49620-85-C-0008 F/G 12/1 NL

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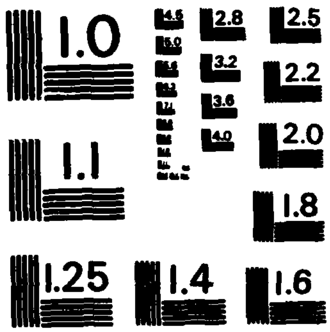

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Non-uniform Bounds of Normal Approximation for Finite-population U-statistics

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Technical keport No. 85-26

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SECURITY CLASSIFICATION OF THIS PAGE

AD A 159163

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 85-0696
6a. NAME OF PERFORMING ORGANIZATION University of Pittsburgh 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research
6c. ADDRESS (City, State and ZIP Code) Pittsburgh, PA 15260	7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F49620-85-C-0008
8c. ADDRESS (City, State and ZIP Code) Bolling AFB DC 20332-6448	10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2304	A5	

11. TITLE (Include Security Classification)
Non-uniform Bounds of Normal Approximation for Finite-population U-statistics

12. PERSONAL AUTHOR(S)
Miao Baiqi and Zhao Lincheng

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Reprint	FROM ______ TO ______	July 1985	35

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	Borel-measurable function
XXXXXXXXXXXX			

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Let A_N be a population with N balls bearing numbers $a_{N1},\dots,a_{NN}$ respectively. Draw n balls from A_N randomly without replacement, and denote the numbers appearing on these n balls by $X_1,\dots,X_n$. Suppose that $\phi_N(x,y)$ be a Borel-measurable function, symmetric in x and y. Set $U_n = \binom{n}{2}^{-1} \sum_{1\le j<k\le n} \phi_N(X_j,X_k)$, $\theta_N = E\phi_N(X_1,X_2)$, $g(X_1) = E(\phi_N(X_1,X_2)|X_1)$, $\sigma_g^2 = Var(g(X_1))$. In this paper we established that, if there exists fixed constants λ_1 and λ_2 such that $0 < \lambda_1 \le n/N < \lambda_2 < 1$, then it is valid for all positive integer n and real x that

$$\left|P\left(\frac{\sqrt{N}(U_n-\theta_N)}{2\sqrt{1-n/N}\,\sigma_g} \le x\right) - \Phi(x)\right| \le C\, n^{-1/2}\sigma_g^{-3}E|\phi_N(X_1,X_2)|^3(1+|x|)^{-3}$$

where $\Phi(x)$ is the standard normal distribution function, and C is an absolute constant depending solely on λ_1 and λ_2.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL Brian W. Woodruff	22b. TELEPHONE NUMBER (Include Area Code) (202) 767-5027 22c. OFFICE SYMBOL NM

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE.

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SECURITY CLASSIFICATION OF THIS PAGE

Abstract

Let A_N be a population with N balls bearing numbers $a_{N1},\ldots,a_{NN}$ respectively. Draw n balls from A_N randomly without replacement, and denote the numbers appearing on these n balls by $X_1,\ldots,X_n$. Suppose that $\phi_N(x,y)$ be a Borel-measurable function, symmetric in x and y. Set $U_n = \binom{n}{2}^{-1} \sum_{1\le j<k\le n} \phi_N(X_j,X_k)$, $\theta_N = E\phi_N(X_1,X_2)$, $g(X_1) = E(\phi_N(X_1,X_2)|X_1)$, $\sigma_g^2 = \mathrm{Var}(g(X_1))$. In this paper we established that, if there exists fixed constants λ_1 and λ_2 such that $0 < \lambda_1 \le n/N < \lambda_2 < 1$, then it is valid for all positive integer n and real x that

$$\left|P\left(\frac{\sqrt{N}(U_n-\theta_N)}{2\sqrt{1-n/N}\,\sigma_g} \le x\right) - \Phi(x)\right| \le C\, n^{-\frac{1}{2}}\sigma_g^{-3}E|\phi_N(X_1,X_2)|^3(1+|x|)^{-3}$$

where $\Phi(x)$ is the standard normal distribution function, and C is an absolute constant depending solely on λ_1 and λ_2.

1. Introduction

Let A_N be a population of N balls bearing real numbers $a_{N1},\dots,a_{NN}$. Draw n balls from A_N randomly without replacement, and denote the numbers appearing on these n balls by $X_1,\dots,X_n$. Suppose that $\phi(x,y) = \phi_N(x,y)$ be a two-variable Borel-measurable function which is symmetric in x and y. Call

$$U_n = \binom{n}{2}^{-1} \sum_{1\le j<k\le n} \phi(X_j,X_k) \tag{1}$$

the finite-population U-statistic with the kernel ϕ. For simplicity and without losing generality, we can assume that $E\phi(X_1,X_2) = 0$. Define $g(X_1) = E(\phi(X_1,X_2)|X_1)$ and suppose that $\sigma_g^2 = Eg^2(X_1) > 0$.

Nandi and Sen (1963) researched the asymptotic normality of U_n. Zhao Lincheng and Chen Xiru (1985) established the ideal Berry-Esseen bounds of U_n under weaker conditions. Considering the profound results about non-uniform convergence rates of U-statistic, established by Zhao Lincheng and Chen Xiru (1983), it is natural to raise such a problem: whether the analogue is true for finite-population U-statistics. But this problem will be more difficult, when $X_1,\dots,X_n$ are not independent.

Recently we studied this problem and established the following main result:

Theorem 1. Suppose that there exist fixed constants λ_1 and λ_2 such that

$$0 < \lambda_1 \le n/N \le \lambda_2 < 1. \tag{2}$$

Then there exists an absolute constant C depending only upon λ_1 and λ_2 such that

$$\left|P\left(\frac{\sqrt{N}U_n}{2\sqrt{1-n/N}\sigma_g} \leq x\right) - \Phi(x)\right| \leq Cn^{-\frac{1}{2}}\sigma_g^{-3}\nu_3(1+|x|)^{-3} \tag{3}$$

for every x and n, where $\nu_3 = E|\phi(X_1,X_2)|^3$ and $\Phi(x)$ is the standard normal distribution function.

II. Some Lemmas

To prove the Theorem 1 we must prove some lemmas in this section. Obviously, we need only to prove the inequality (3) for large n and all real x. For convenience, we often omit the index N and the phrase "for large n". Besides, without losing generality, we can suppose that $\sigma_g = 1$. Let I(A) denote the indicator of set A,#(A) denote the number of different elements in set A, and the letter i denote $\sqrt{-1}$ especially. Last, for simplicity of presentation, we make the following conventions:

1. In this paper, "absolute constant" means the positive constant depending only upon λ_1 and λ_2, which is independent of n, N, A_N, and ϕ, and can assume different values on each of its appearance even within the same formula. Throughout this paper, we will use $C, C^*, C', C'', \lambda, \mu, \varepsilon, \varepsilon^*$, etc. for some absolute constants, use $Q_1(|t|)$, $Q_2(|\psi|)$ and $Q_3(|\psi|,|t|)$ for some polynomials with absolute constant coefficients. Further these polynomials can also take different forms on each of their appearance.

2. Set

$$b_j = g(a_{Nj})/\sqrt{N},\ L_N = \sum_{j=1}^{N} |b_j|^3.$$

Let $\psi_1(t)$ and $\psi_2(t)$ be two functions (which may depend on n) defined on R^1. We call $\psi_1 \sim \psi_2$, if there exists an absolute constant λ such that

$$\int_{|t|\leq\lambda L_N^{-1}} |t|\,|\psi_1(t) - \psi_2(t)|dt \leq CN^{-\frac{1}{2}}\nu_3. \qquad (4)$$

In this paper, the following symbols are often used: p = n/N, q = 1-p, (p,q are both dependent on n and N)

$$\eta_j = g(X_j)/\sqrt{N}, \quad \xi_j = g(X_j)/\sqrt{Npq}, \quad S_n = \sum_{j=1}^{n} \xi_j, \quad S_n' = \sum_{j=1}^{J} \xi_j, \quad S_n'' = S_n - S_n',$$

where $J < n$ is to be defined. Write

$$g_j = g(X_j), \quad \phi_{jk} = \phi(X_j, X_k), \; j \neq k,$$

$$Y_{jk} = \phi_{jk} - \frac{N-1}{N-2}(g_j + g_k), \; 1 \leq j \neq k \leq n$$ (The j,k's range is the same in the following). Set

$$\hat{\phi}_{jk} = \phi_{jk} I(|\phi_{jk}| \leq \sqrt{N}), \quad \phi_{jk}^* = \hat{\phi}_{jk} - E\hat{\phi}_{jk},$$

$$g_j^* = E(\phi_{jk}^* | X_j), \quad Y_{jk}^* = \phi_{jk}^* - \frac{N-1}{N-2}(g_j^* + g_k^*),$$

$$\Delta_n = \frac{N-2}{N-1} \cdot \frac{\sqrt{n}}{2\sqrt{g}} \binom{n}{2}^{-1} \sum_{1 \leq k < l \leq n} Y_{jk} \stackrel{\Delta}{=} d_n \sum_{1 \leq j < k \leq n} Y_{jk},$$

$$\Delta_n^* = d_n \sum_{1 \leq j < k \leq n} Y_{jk}^*, \; \Delta_{n1}^* = d_n \sum_{1 \leq j < k \leq J} Y_{jk}^*,$$

$$\Delta_{n2}^* = \Delta_n - \Delta_{n1}^* = d_n \sum_{k=J+1}^{n} \sum_{j=1}^{k-1} Y_{jk}^*,$$

where $d_n = 0(n^{-3/2})$. Let

$$\tilde{U}_n = S_n + \Delta_n = \frac{N-2}{N-1} \cdot \frac{\sqrt{n} Un}{2\sqrt{q}} .$$

It is obvious that

$$\sum_{j=1}^{N} b_j = 0, \; \sum_{j=1}^{N} b_j^2 = 1, \; E\eta_1^2 = 1/N,$$

and

$$1/N \leq L_N = \frac{1}{\sqrt{N}} E|g(X_1)|^3 \leq \nu_3/\sqrt{N},$$

$$E\, S_n = 0,\ \mathrm{Var}(S_n) = N/(N-1).$$

Suppose that $\{j_1,\ldots,j_k\} \subset \{3,4,\ldots,n\}$. It is easy to see that

$$E(Y_{12}|X_j) = 0, \qquad \text{for } j = 1,2,\ldots,N, \tag{5}$$

$$E(Y_{12}|X_1, X_{j_1},\ldots,X_{j_k}) = -\frac{1}{N-k-1}\sum_{\ell=1}^{k} Y_{1j_\ell}, \tag{6}$$

$$E(Y_{12}|X_{j_1},\ldots,X_{j_k}) = \binom{N-k}{2}^{-1} \sum_{1\leq \ell < m \leq k} Y_{j_\ell j_m}, \text{for } k \geq 2. \tag{7}$$

<u>Lemma 1</u>. For any $\alpha > 0$ and any $n \leq N$, we have

$$E|S_n|^\alpha \leq C = C(\alpha).$$

<u>Proof</u>. We only prove lemma 1 for every even natural number $2k$, that is $E|S_n|^{2k} \leq C(k)$.

Because

$$E\left(\sum_{j=1}^{n} \eta_j\right)^{2k} = \sum_{m=1}^{2k} \sum{}' \frac{(2k)!}{r_1!\ldots r_m!} \binom{n}{m} E(\eta_1^{r_1}\ldots\eta_m^{r_m}),$$

here the summation $\sum'$ is carried out over all integers $r_1,\ldots,r_m$ satisfying $r_1+\ldots+r_m = 2k$ and $r_1 \geq 1,\ldots,r_m \geq 1$. If some $r_j = 1$, for example $r_m = 1$, then we have

$$\binom{n}{m} E(\eta_1^{r_1}\ldots\eta_m^{r_m}) = -\frac{1}{N-m+1}\binom{n}{m}\sum_{j=1}^{m-1} E(\eta_j \eta_1^{r_1}\ldots\eta_{m-1}^{r_{m-1}}),$$

from $E(\eta_m|\eta_1,\ldots,\eta_{m-1}) = -\frac{1}{N-m+1}\sum_{j+1}^{m-1} \eta_j$. So the contribution of this term

to $E(\sum_{j=1}^{n}\eta_j)^{2k}$ can be merged into some summands with the forms $\binom{n}{m-1}E(\eta_1^{r_1'}\ldots\eta_{m-1}^{r_{m-1}'})$ and does not change the orders of magnitude of these summands. Hence in the expansion of $E(\sum_{j=1}^{n}\eta_j)^{2k}$ the terms with some $r_j = 1$ can be omitted, and we get

$$E(\sum_{j=1}^{n}\eta_j)^{2k} \leq \sum_{m=1}^{k} \sum{}'' C(k)\binom{n}{m}|E(\eta_1^{r_1}\ldots\eta_m^{r_m})|,$$

here the summation $\sum''$ is carried out over all integers $r_1,\ldots,r_m$ satisfying $r_1+\ldots+r_m = 2k$ and $r_1 \geq 2,\ldots,r_m \geq 2$. In this case $\binom{n}{m}E|\eta_1^{r_1}\ldots\eta_m^{r_m}| \leq C(k)$ so $E(\sum_{j=1}^{n}\eta_j)^{2k} \preceq C(k)$, and the lemma is proved from (2).

<u>Lemma 2.</u> For any $n \leq N$, we have

$$E(\sum_{1\leq j<k\leq n} Y_{jk})^4 = Cn^3EY_{12}^4 + Cn^4(EY_{12}^2)^2.$$

<u>Proof.</u> Write $W_n = \sum_{1\leq j<k\leq n} Y_{jk}$. In the expansion of $E\,W_n^4$, we need not take account of those terms, in which some index only appears one time. As an example, we consider those terms such as $E\,Y_{j_1j_2}Y_{j_1j_3}Y_{j_2j_4}Y_{j_3j_5}$, where $j_1,\ldots,j_5$ are different and the index j_5 is single. From (6) we have

$$E(Y_{j_1j_2}Y_{j_1j_3}Y_{j_2j_4}Y_{j_3j_5}) = E\{Y_{12}Y_{13}Y_{24}E(Y_{35}|X_1,X_2,X_3,X_4)\}$$

$$= -\frac{1}{N-4}E\,Y_{12}Y_{13}Y_{24}(Y_{13}+Y_{23}+Y_{34}).$$

Since the number of such terms do not exceed Cn^5, the contributions of these terms to $E\,W_n^4$ can be merged into the terms with 4 indexes and don't change the orders of magnitude of the latter. Using Schwarz's inequality, we get, for example, $|EY_{12}Y_{23}Y_{34}Y_{14}| \le E(Y_{12}^2Y_{34}^2)$. So

$$E\,W_n^4 \le Cn^3EY_{12}^4 + Cn^4E(Y_{12}^2Y_{34}^2) \le Cn^3EY_{12}^4 + Cn^4(EY_{12}^2)^2,$$

from $d_n = O(n^{-3/2})$, the lemma is proved.

Lemma 3. Without (2) but with the condition $I \triangleq I_n = n - J \le J \triangleq J_n$ and $J/(N-n+1) \le \lambda$, we have

$$E\left|\sum_{k=2}^{n} Y_{1k}\right|^3 \le Cn^{3/2}E|Y_{12}|^3, \tag{8}$$

$$E\left|\sum_{1\le j<k\le n} Y_{jk}\right|^3 \le Cn^3E|Y_{12}|^3, \tag{9}$$

$$E\left|\sum_{j=1}^{J}\sum_{k=J+1}^{n} Y_{jk}\right|^3 \le C(\lambda)(IJ)^{3/2}E|Y_{12}|^3, \tag{10}$$

where $C(\lambda)$ is a constant depending only upon λ.

Proof. The proofs of these inequalities are similar, so we only prove (9). Set

$$\xi_k = \sum_{j=1}^{k-1} Y_{jk}, \quad W_n = \sum_{k=2}^{n} \xi_k,$$

then

$$E|W_n|^3 = E(\xi_n^2|W_n|) + E(W_{n-1}^2|W_n|) + 2E(\xi_n W_{n-1}|W_n|)$$

$$\triangleq \sum_{\ell=1}^{3} M_\ell,$$

$$M_3 \leq 2E\{(\xi_n W_{n-1}|\xi_n| + \xi_n W_{n-1}|W_{n-1}|)I(\xi_n W_{n-1} \geq 0)\}$$

$$+ 2\ E\{(\xi_n W_{n-1}|W_{n-1}| - \xi_n W_{n-1}|\xi_n|)I(\xi_n W_{n-1}<0)\}$$

$$= 2\ E(\xi_n W_{n-1}|W_{n-1}|) + 2\ E(\xi_n^2|W_{n-1}|).$$

From (6), we have $E(\xi_n|X_1,\ldots,X_{n-1}) = \sum_{j=1}^{n-1} E(Y_{jn}|X_1,\ldots,X_{n-1}) = -\frac{2}{N-n+1} W_{n-1}$,

so

$$M_3 \leq 2\ E(\xi_n^2|W_{n-1}|) \leq 2\ E(\xi_n^2|W_n|) + 2\ E|\xi_n|^3,$$

and

$$E|W_n|^3 \leq 3\ E(\xi_n^2|W_n|) + (\tfrac{2}{3}\ E|W_{n-1}|^3 + \tfrac{1}{3}\ E|W_n|^3) + 2\ E|\xi_n|^3,$$

$$E|W_n|^3 \leq \frac{9}{2}\ (E|\xi_n|^3)^{2/3}(E|W_n|^3)^{1/3} + E|W_{n-1}|^3 + 3\ E|\xi_n|^3.$$

Set

$$y_n = E|W_n|^3,\ a = \sup_{2\leq k\leq n} E|\xi_k|^3,$$

then by above last inequality we obtain

$$y_n \leq \frac{9}{2}\ a^{2/3} y_n^{1/3} + y_{n-1} + 3a,$$

$$y_{n-1} \leq \frac{9}{2}\ a^{2/3} y_{n-1}^{1/3} + y_{n-2} + 3a,$$

$$\cdots\cdots$$

So

$$y_n \leq \frac{9}{2} a^{2/3}(y_n^{1/3}+y_{n-1}^{1/3}+\ldots+y_2^{1/3}) + 3na,$$

$$y_{n-1} \leq \frac{9}{2} a^{2/3}(y_{n-1}^{1/3}+y_{n-2}^{1/3}+\ldots+y_2^{1/3}) + 3na,$$

....

Define $y = \sup\limits_{2\leq k\leq n} y_k$, then

$$y \leq \frac{9}{2} na^{2/3}y^{1/3} + 3na$$

From this estimate, we get

$$E|W_n|^3 \leq Cn^{3/2} \sup_{2\leq k\leq n} E|\xi_k|^3,$$

and (9) is obtained from (8).

Lemma 4. Let ε and ε^* be any fixed positive numbers, and $J \leq n$. Set

$$A_J = \{(X_1,\ldots,X_J): \sum_{j=1}^{J} \eta_j^2 \geq J/N+\varepsilon\}, \tag{11}$$

$$B_J = \{(X_1,\ldots,X_J): |\sum_{j=1}^{J} \eta_j| \geq \varepsilon^* L_N^{-1}\}, \tag{12}$$

then under the condition of the Theorem 1, the following estimate is valid:

$$P(A_J \cup B_J) \leq CL_N^2.$$

The proof of this lemma is almost the same as lemma 1 of the paper [5].

Suppose $J \geq 0, \mu_1 N^{1-\alpha} \leq I \overset{\Delta}{=} n-J \leq \mu_2 N^{1-\alpha}, \mu_1, \mu_2 > 0, \alpha \in [0, \frac{1}{2}]$. Set $\tilde{N} = N-J$, $\tilde{p} = I/\tilde{N}$, $\tilde{q} = 1-\tilde{p}$. It is obvious that

$$\tilde{p} = (n-J)/(N-J) \leq n/N \leq \lambda_2 < 1.$$

Let $C^* > 1$, $C', C'' > 0$ and $\{j_1, \ldots, j_J\} \subset \{1, \ldots, N\}$. Define

$$D_N = \{j: 1 \leq j \leq N, \ |b_j| > C^* L_N\}, \tag{13}$$

$$G_J = \{1, \ldots, N\} - \{j_1, \ldots, j_J\}, \tag{14}$$

$$\tilde{\xi}_k = \tilde{N}^{-\frac{1}{2}}\psi + tb_k, \ \tilde{\omega}_k = \tilde{\xi}_k/\sqrt{\tilde{p}\tilde{q}}, \quad \delta_k = \delta_k(\psi, t) = \tilde{q} + \tilde{p}e^{i\tilde{\omega}_k}, \tag{15}$$

$$\xi_k = N^{-\frac{1}{2}}\psi + tb_k, \ \omega_k = \xi_k/\sqrt{pq}, \quad \rho_k = \rho_k(\psi, t) = qe^{-ip\omega_k} + pe^{iq\omega_k}, \tag{16}$$

$$\tilde{\Gamma}_1 = \{(\psi, t): |\psi| \leq 2C'\sqrt{\tilde{N}\tilde{p}\tilde{q}}, \ |t| \leq C''\sqrt{\tilde{p}\tilde{q}}\ b_*^{-1}\}, \ b_* = \max_{1 \leq k \leq n} |b_k|,$$

$$\tilde{\Gamma}_2 = \{(\psi, t): 2C'\sqrt{\tilde{N}\tilde{p}\tilde{q}} \leq |\psi| \leq \pi\sqrt{\tilde{N}\tilde{p}\tilde{q}}, \ |t| \leq C''\sqrt{\tilde{p}\tilde{q}}\ b_*^{-1}\},$$

$$\tilde{\Gamma}_3 = \{(\psi, t): |\psi| \leq 2C'\sqrt{\tilde{N}\tilde{p}\tilde{q}}, \ C''\sqrt{\tilde{p}\tilde{q}}\ b_*^{-1} \leq |t| \leq C''\sqrt{\tilde{p}\tilde{q}}\ L_N^{-1}\},$$

$$\tilde{\Gamma}_4 = \{(\psi, t): 2C'\sqrt{\tilde{N}\tilde{p}\tilde{q}} \leq |\psi| \leq \pi\sqrt{\tilde{N}\tilde{p}\tilde{q}}, \ C''\sqrt{\tilde{p}\tilde{q}}\ b_*^{-1} \leq |t| \leq C''\sqrt{\tilde{p}\tilde{q}}\ L_N^{-1}\}.$$

In the definition of $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_4$, taking off all the "~" we get new sets of (ψ, t) and denote these sets by $\Gamma_1, \ldots, \Gamma_4$.

Suppose

$$\sum_{\ell=1}^{J} b_{j_\ell}^2 \leq J/N + \varepsilon, \left| \sum_{\ell=1}^{J} b_{j_\ell} \right| \leq \varepsilon^* L_N^{-1}, \text{ for } \varepsilon, \varepsilon^* > 0. \tag{17}$$

<u>Lemma 5</u>. Let positives ε and ε^* be small and C^* be large, C', C'' be small enough, and $C' \geq C''C^*$. Then, when (17) is valid and $(\psi,t)\varepsilon\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 \cup \tilde{\Gamma}_3 \cup \tilde{\Gamma}_4$, there exist absolute constants C and μ such that

$$\Big|\prod_{k\varepsilon G_J-D_N} \delta_k(\psi,t)\Big| \leq C\exp\{-\mu(\psi^2+t^2)\}, \tag{18}$$

$$\Big|\prod_{k\varepsilon G_J-D_N-\Lambda} \delta_k(\psi,t)\Big| \leq \begin{cases} CN^{-3}\exp\{-\mu(\psi^2+t^2)\}, & \text{when } (\psi,t)\varepsilon\tilde{\Gamma}_2\cup\tilde{\Gamma}_4 \quad (19) \\ CL_N^6\exp\{-\mu(\psi^2+t^2)\}, & \text{when } (\psi,t)\varepsilon\tilde{\Gamma}_3, \quad (20) \end{cases}$$

for N large enough, where set $\Lambda = \{m_1,m_2,m_3,m_4\}$ is any subset of $\{1,\ldots,N\}$.

<u>Proof</u>. The proof of (18) can be referred to the lemma 2 of the paper[5]. From (18) the estimate (19) also be deduced. Now suppose that $(\psi,t)\varepsilon\tilde{\Gamma}_3$, using (18), we get

$$\Big|\prod_{k\varepsilon G_J-D_N} \delta_k(\psi,t)\Big| \leq CL_N^6\exp\{-\mu(\psi^2+t^2)\}. \tag{21}$$

If some element m of Λ does not belong to D_N, then

$$|\tilde{\omega}_m| \leq \frac{1}{\sqrt{\tilde{p}\tilde{q}}}\left(|\psi\tilde{N}^{-1/2}|+|tb_k|\right) \leq \left(2C'\sqrt{\tilde{N}\tilde{p}\tilde{q}}\,\tilde{N}^{-1/2}+C''\sqrt{\tilde{p}\tilde{q}}\,L_N^{-1}C^*L_N\right)/\sqrt{\tilde{p}\tilde{q}}$$

$$\leq 2C' + C''C^* \triangleq \tilde{C}.$$

Taking $\tilde{C}$ small enough, from the inequality $1-\cos\tilde{\omega}_m \leq 1-\cos\tilde{C} \leq \tilde{C}^2/2$, we have

$$|\rho_m(\psi,t)|^2 = 1 - 2\tilde{p}\tilde{q}(1-\cos\tilde{\omega}_m) > 1 - \tilde{c}^2\tilde{p}\tilde{q} > 1 - \tilde{c}^2 > 0.$$

Using this inequality and (21) we proved (20).

Lemma 6. *Suppose (2) is valid and $0 < \mu_1 N^{-\alpha} \le (n-J)/N \le \mu_2 N^{-\alpha}$ for $\alpha \in [0,\frac{1}{2}]$* and select ε and ε^* appropriately, then there exist λ and μ such that

$$|E\{(S_n'')^r e^{itS_n''} X_1,\ldots,X_J\}| \le CN^{-r\alpha/2}(1+|S_n'|^r + N^{-r\alpha/2}|t|^r)\cdot$$
$$\cdot \exp\{-\mu N^{-\alpha} t^2\}, \quad \text{for } r = 0,1,2, \tag{22}$$

$$|E\{(S_n'')^3 e^{itS_n''}|X_1,\ldots,X_J\}| \le CN^{-3\alpha/2}(1+N^{\alpha/2}L_N + |S_n'|^3 + N^{-3\alpha/2}|t|^3)$$
$$\cdot \exp\{-\mu N^{-\alpha} t^2\}, \tag{23}$$

provided $|t| \le \lambda L_N^{-1}\sqrt{pq}$ and $(X_1,\ldots,X_J) \in A_J^c \cap B_J^c$.

Proof. The case of $r = 0$ in (22) can be proved by lemma 2 of the paper [5]. In other cases, let $X_\ell = a_{Nj_\ell}$, $\ell = 1,\ldots,J$, and write $\tilde{S}_n'' = (\tilde{p}\tilde{q})^{-\frac{1}{2}} \sum_{j \in G_J} \eta_j$.

Because of the releationship between S_n'' and $\tilde{S}_n''$, we only need to prove that

$$M_1 \triangleq |E(\tilde{S}_n'' e^{it\tilde{S}_n''}|X_1,\ldots,X_J)| \le C(1+|S_n'|+|t|)\exp\{-\mu t^2\}, \tag{24}$$

$$M_2 \triangleq |E\{(\tilde{S}_n'')^2 e^{it\tilde{S}_n''}|X_1,\ldots,X_J\}| \le C(1+|S_n'|^2+t^2)\exp\{-\mu t^2\}, \tag{25}$$

$$M_3 \triangleq |E\{(\tilde{S}_n'')^3 e^{it\tilde{S}_n''}|X_1,\ldots,X_J\}| \le C(1+N^{\frac{\alpha}{2}}L_N+|S_n'|+|t|^3)\exp\{-\mu t^2\}, \tag{26}$$

where $|t| \le \lambda\sqrt{\tilde{p}\tilde{q}}\, L_N^{-1}$.

Define $B_I(\tilde{p}) = \sqrt{2\pi}\binom{\tilde{N}}{I}\tilde{p}^I\tilde{q}^{\tilde{N}-I}$. It is not difficult to see that

$$E(e^{it\tilde{S}''_n} \mid X_1,\ldots,X_J)$$

$$= \frac{1}{B_I(\tilde{p})} \cdot \frac{1}{\sqrt{2\pi}} \int_{|\theta|\le\pi} \prod_{k\in G_J} (\tilde{q}+\tilde{p}\exp\{i(tb_k/\sqrt{\tilde{p}\tilde{q}}+\theta)\})e^{-iI\theta}d\theta$$

$$= \frac{1}{\sqrt{N\tilde{p}\tilde{q}}B_I(\tilde{p})} \cdot \frac{1}{\sqrt{2\pi}} \int_{|\psi|\le\pi\sqrt{N\tilde{p}\tilde{q}}} \{\prod_{k\in G_J} \delta_k(\psi,t)\}\exp\{-iI\psi/\sqrt{N\tilde{p}\tilde{q}}\}d\psi \tag{27}$$

from the equality

$$\int_{-\pi}^{\pi} e^{ik\theta}d\theta = \begin{cases} 2\pi & \text{for integer } k = 0, \\ 0 & \text{for integer } k \neq 0. \end{cases}$$

Differentiating both sides of (27), we obtain

$$M_1 \le C\int_{|\psi|<\pi\sqrt{N\tilde{p}\tilde{q}}} T_1 d\psi \triangleq C\int_{|\psi|<\pi\sqrt{N\tilde{p}\tilde{q}}} \Big|\sum_{k\in G_J} \frac{\tilde{p}}{\sqrt{\tilde{p}\tilde{q}}} b_k e^{i\tilde{\omega}_k} \prod_{\substack{j\in G_J \\ j\neq k}} \delta_j\Big| d\psi \tag{28}$$

here we have already used Striling's formula to get the following estimate:

$$\sqrt{N\tilde{p}\tilde{q}}B_I(\tilde{p}) = 1+O(N^{-(1-\alpha)}), \text{ for } \alpha\in[0,\tfrac{1}{2}] \tag{29}$$

Take C' and C" small enough such that $|\tilde{\omega}_k|<1/8$ for $(\psi,t)\in\Gamma_1$ and $k \in G_J$. In this case, we have

$$e^{i\tilde{\omega}_k}/\delta_k = 1 + \theta_k\tilde{\omega}_k .$$

Here and after, θ_k can be assumed different values and $|\theta_k| \le C$. We have

$$|\sum_{k\varepsilon G_J} \frac{\tilde{p}b_k}{\sqrt{\tilde{p}\tilde{q}}\delta_k} e^{i\tilde{\omega}k}| = |\sum_{k\varepsilon G_J} \frac{\tilde{p}b_k}{\sqrt{\tilde{p}\tilde{q}}} (1+\frac{\theta_k\tilde{\xi}_k}{\sqrt{\tilde{p}\tilde{q}}})|$$

$$\le C|\sum_{k\varepsilon G_J} b_k| + C\sum_{k\varepsilon G_J} |b_k\tilde{\xi}_k| \le C|\sum_{\ell=1}^{J} b_{j_\ell}| + C|\psi|N^{-\frac{1}{2}}\sum_k |b_k| + C|t|$$

$$\le C\{|\sum_{\ell=1}^{J} b_{j_\ell}| + (|\psi|+|t|)\}.$$

Thus, it is inferred by (28) and lemma 5 that

$$T_1 \le C\{|\sum_{\ell=1}^{J} b_{j_\ell}| + |\psi| + |t|\}\exp\{-\mu(t^2+\psi^2)\}, \text{ for } (\psi,t)\varepsilon\tilde{\Gamma}_1, \tag{30}$$

$$T_1 \le C \sum_k |b_k| \prod_{j\varepsilon G_J, j\neq k} |\delta_j(\psi,t)| \le C \sum_k |b_k| \cdot CN^{-1}\exp\{-\mu(\psi^2+t^2)\}$$

$$\le C \exp\{-\mu(\psi^2+t^2)\}, \text{ for } (\psi,t)\varepsilon\tilde{\Gamma}_2 \cup \tilde{\Gamma}_4. \tag{31}$$

Now consider the case $(\psi,t)\varepsilon\tilde{\Gamma}_3$. From $\sum_{j=1}^{N} b_j = 0$, we have

$$|\sum_{j\varepsilon G_J - D_N} b_j| \le |\sum_{\ell=1}^{J} b_{j_\ell}| + \sum_{j\varepsilon D_N} |b_j|, \tag{32}$$

$$\sum_{j\varepsilon D_N} |b_j| \le C^{*-1}L_N^{-1} \sum_{j\varepsilon D_N} b_j^2 \le (C^* L_N)^{-1}. \tag{33}$$

Take C' and C" appropriately small such that $|\tilde{\omega}_k| < 1/8$, where $k\varepsilon G_J - D_N$, then $e^{i\tilde{\omega}k}/\delta_k = 1 + \theta_k|\tilde{\xi}_k|/\sqrt{\tilde{p}\tilde{q}}$. Using (20), (32) and (33), we get

$$T_1 \le \left| \sum_{k\varepsilon G_J - D_N} \frac{\tilde{p} b_k}{\sqrt{\tilde{p}\tilde{q}}\,\delta_k} e^{i\tilde{\omega}k} \prod_{j\varepsilon G_J} \delta_j \right| + C \sum_{k\varepsilon D_N} |b_k| \prod_{j\varepsilon G_J, j\neq k} |\delta_j|$$

$$\le C \left| \sum_{k\varepsilon G_J - D_N} b_k \right| \prod_{j\varepsilon G_J} |\delta_j| + C \sum_{k\varepsilon G_J - D_N} |b_k \tilde{\xi}_k| \prod_{j\varepsilon G_J} |\delta_j|$$

$$+ C \sum_{k\varepsilon D_N} |b_k| \prod_{j\varepsilon G_J, j\neq k} |\delta_j|$$

$$\le C \{ (| \sum_{\ell=1}^{J} b_{j_\ell} | + (C^* L_N)^{-1}) + C(|\psi| + |t|) + (C^* L_N)^{-1} \} \cdot$$

$$\cdot\ L_N \exp\{-\mu(\psi^2 + t^2)\}$$

$$\le C(| \sum_{\ell=1}^{J} b_{j_\ell} | + 1 + |\psi| + |t|) \exp\{-\mu(\psi^2 + t^2)\}. \tag{34}$$

So (24) can be inferred by (28), (30), (31) and (34). It is similar to get (25). In order to prove (26), differentiating three times in the both sides of (27), we have

$$M_3 \le C \int_{|\psi| \le \pi\sqrt{N\tilde{p}\tilde{q}}} T_3 d\psi$$

where $T_3 = T_3' + T_3'' + T_3'''$, and

$$T_3' = \tilde{p}(\tilde{p}\tilde{q})^{-3/2} \left| \sum_{k\varepsilon G_J} b_k^3\, e^{i\tilde{\omega}k} \frac{1}{\delta_k} \prod_{m\varepsilon G_J} \delta_m \right|$$

$$T_3'' = 3\tilde{p}^2(\tilde{p}\tilde{q})^{-3/2}\Big|\sum_{\substack{k,j\in G_J\\ k\neq j}} b_k^2 b_j e^{i(\tilde{\omega}_k+\tilde{\omega}_j)} \prod_{\substack{m\in G_J\\ m\neq k,j}} \delta_m\Big|,$$

$$T_3''' = \tilde{p}^3(\tilde{p}\tilde{q})^{-3/2}\Big|\sum_{\substack{k,j,\ell\in G_J\\ k\neq j\neq \ell\neq k}} b_k b_j b_\ell e^{i(\tilde{\omega}k+\tilde{\omega}j+\tilde{\omega}\ell)} \prod_{\substack{m\in G_J\\ m\neq k,j,\ell}} \delta m\Big|$$

The estimate of each term is similar to (22), but the term $c\, N^{\frac{\alpha}{2}}L_N\, e^{-\mu t^2}$ appears in the right side of (26). The reason is that if C' and C" are small enough, we obtain by lemma 5 that

$$T_3' \le CN^{\frac{\alpha}{2}}\sum_{k\in G_J}|b_k|^3\prod_{m\in G_J}|\delta_m| \le CN^{\frac{\alpha}{2}}L_N\exp\{-\mu(\psi^2+t^2)\} .$$

The other estimates are omitted. The lemma 6 is proved.

Lemma 7. Under the condition of the Theorem 1, there exists $\lambda > 0$ such that

$$\int_{|t|<\lambda L_N^{-1}} |t|^{-1}|i^3ES_n^3e^{itS_n} - \psi_n(t)|dt \le CL_N,$$

where $\psi_n(t) = (i^3E\, S_n^3 + 3t - t^3)e^{-\frac{t^2}{2}}$.

Proof. Set

$$u_k = (-e^{-ip\omega k}+ e^{iq\omega k})/\rho_k \overset{\Delta}{=} \hat{u}_k/\rho_k,$$

$$v_k = (pe^{-ip\omega k}+qe^{iq\omega k})/\rho_k \overset{\Delta}{=} \hat{v}_k/\rho_k,$$

$$w_k = (-p^2e^{-ip\omega k}+q^2e^{iq\omega k})/\rho_k \overset{\Delta}{=} \hat{w}_k/\rho_k,$$

$$B_n(p) = \sqrt{2\pi}\binom{N}{n}p^nq^{N-n}.$$

From the equality

$$E\, e^{itS_n} = \frac{1}{\sqrt{Npq}\; B_n(p)} \cdot \frac{1}{\sqrt{2\pi}} \int_{|\psi| \leq \pi\sqrt{Npq}} \prod_{k=1}^{N} \rho_k(\psi,t)\, d\psi, \qquad (35)$$

we have

$$(E\, e^{itS_n})''' = \frac{1}{\sqrt{Npq}\; B_n(p)} \cdot \frac{1}{\sqrt{2\pi}} \int_{|\psi| \leq \pi\sqrt{Npq}} (T_1+T_2+T_3) \prod_{k=1}^{N} \rho_k(\psi,t)\, d\psi, \qquad (36)$$

here

$$T_1 = -\frac{i}{\sqrt{pq}} \sum_{k=1}^{N} b_k^3 w_k, \qquad (37)$$

$$T_2 = -3i\sqrt{pq} \sum_{1 \leq k \neq j \leq N} b_k^2 v_k b_j u_j \qquad (38)$$

$$T_3 = -i(pq)^{3/2} \sum_{\substack{1 \leq k,j,\ell \leq N \\ k \neq j \neq \ell \neq k}} b_k b_j b_\ell u_k u_j u_\ell, \qquad (39)$$

Take C' and C" small enough to satisfy $|\omega_k| < 1/10$ for $(\psi,t)\in\Gamma_1$. In this case,

$$u_k = i\omega_k[1+\frac{1}{2}\, i(q-p)\omega_k+\theta_k\xi_k^2], \qquad (40)$$

$$v_k = 1 + i(q-p)\omega_k + \theta_k\xi_k^2, \qquad (41)$$

$$w_k = (q-p) + \theta_k|\xi_k|. \qquad (42)$$

Substituting (42) into (37), we obtain that, when $(\psi,t)\in\Gamma_1$,

$$\left| T_1 - \frac{i(p-q)}{\sqrt{pq}} \sum_{k=1}^{N} b_k^3 \right| \leq C \sum_{k=1}^{N} |b_k^3 \xi_k| \leq C \sum_k |b_k|^3 (N^{-\frac{1}{2}}|\psi| + |tb_k|)$$

$$\leq C\, L_N (N^{-\frac{1}{2}}|\psi| + |t|). \qquad (43)$$

Further, substituting (40) and (41) into (38), we get

$$T_2 = 3\sum_{1\le k\ne j\le N} b_k^2 b_j \xi_k + 3\sum_{1\le k\ne j\le N} b_k^2 b_j \xi_j \Big[\frac{i(q-p)}{2\sqrt{pq}}(2\xi_k+\xi_j)$$

$$+ \theta_j\theta_k(\xi_k^2+\xi_j^2)\Big] \stackrel{\Delta}{=} T_2' + T_2''.$$

where

$$T_2' = 3\sum_{1\le k\ne j\le N} b_k^2 b_j (N^{-\frac{1}{2}}\psi + t b_j) = 3t\sum_{k=1}^{N} b_k^2(1-b_k^2) - 3N^{-\frac{1}{2}}\psi\sum_{k=1}^{N} b_k^3,$$

$$|T_2''| \le C\Big|\sum_{1\le k\ne j\le N} b_k^2 b_j \xi_k \xi_j\Big| + C\Big|\sum_{k=1}^{N} b_k^2 \sum_{j=1, j\ne k} b_j \xi_j^2\Big|$$

$$+ C\sum_{1\le k\ne j\le N}(b_k^2\xi_k^2|b_j\xi_j| + b_k^2|b_j\xi_j^3|).$$

Using this estimates, we can easily get

$$|T_2 - 3t| \le C\, L_N^2 Q_2(|\psi|) + C\, L_N|t|Q_3(|\psi|,|t|), \quad (44)$$

provided $(\psi,t)\in\Gamma_1$.

Substituting (40) into (39), we have

$$T_3 = -\sum_{k\ne j\ne \ell\ne k} b_k b_j b_\ell \xi_k \xi_j \xi_\ell \Big(1+\frac{i(q-p)}{2\sqrt{pq}}\xi_k + \theta_k\xi_k^2\Big)\cdot$$

$$\cdot\Big(1+\frac{i(q-p)}{2\sqrt{pq}}\xi_j + \theta_j\xi_j^2\Big)\Big(1+\frac{i(q-p)}{2\sqrt{pq}}\xi_\ell + \theta_\ell\xi_\ell^2\Big).$$

Using an argument similar to above, we get

$$|T_3 - t^3| \le C\, L_N^2 Q_2(|\psi|) + C\, L_N|t|Q_3(|\psi|,|t|), \text{ for } (\psi,t)\in\Gamma_1, \quad (45)$$

When $(\psi,t)\varepsilon\Gamma_1$, from (39) of the paper[5], there exists θ such that $|\theta| \leq C$ and

$$\prod_{k=1}^{N} \rho_k(\psi,t) = e^{-\frac{1}{2}(\psi^2+t^2)} + \theta L_N(|\psi|^3+|t|^3)e^{-\frac{1}{4}(\psi^2+t^2)}. \quad (46)$$

Noticing $i^3E\ S_n^3 = \frac{i(p-q)}{\sqrt{pq}} \sum_{k=1}^{N} b_k^3(1+O(\frac{1}{N}))$ and using above estimates, we get

$$(T_1 + T_2 + T_3)\sum_{k=1}^{N} \rho_k(\psi,t) = (i^3E\ S_n^3+3t-t^3)e^{-\frac{1}{2}(\psi^2+t^2)} +$$

$$+ \theta[L_N^2Q_2(|\psi|) + L_N|t|Q_3(|\psi|,|t|)]e^{-\frac{1}{4}(\psi^2+t^2)}, \text{ for } (\psi,t)\varepsilon\Gamma_1. \quad (47)$$

When $(\psi,t)\varepsilon\Gamma_2 \cup \Gamma_4$, from (19), we have

$$|(T_1 + T_2 + T_3)\sum_{k=1}^{N} \rho_k| \leq C \sum_{k=1}^{N} |b_k|^3 \prod_{m=1,m\neq k}^{N} |\rho_m| + C\sum_{1\leq k\neq j\leq N} b_k^2|b_j| \prod_{m=1,m\neq k,j}^{N} |\rho_m|$$

$$+ C\sum_{1\leq k\neq j\neq \ell\neq k\leq N} |b_kb_jb_\ell| \sum_{m=1,m\neq k,j,\ell}^{N} |\rho_m|$$

$$\leq C(L_N+N^{\frac{1}{2}}+N^{\frac{3}{2}})N^{-3}\exp\{-\mu(\psi^2+t^2)\}.$$

Noticing that $(\psi,t)\varepsilon\Gamma_2 \cup \Gamma_4$ implies $|\psi| \geq 2C'\sqrt{Npq}$, we have

$$|(i^3ES_n^3+3t-t^3)e^{-\frac{1}{2}(\psi^2+t^2)}| \leq C\{L_N^2+(|t|+|t|^3)L_N\}\exp\{-\mu(\psi^2+t^2)\},$$

so the estimate (47) is also valid for $(\psi,t)\varepsilon\Gamma_2 \cup \Gamma_4$.

When $(\psi,t)\varepsilon\Gamma_3$, the case is more complicated. Set $H_N = \{1,2,\ldots,N\} - D_N$. It is obvious that $|\omega_k| \leq 2C' + C''C^*$ for $k\varepsilon H_N$. After fixing C^*, we can take C' and C'' small enough such that $|u_k| \leq C|\xi_k|$ for $k\varepsilon H_N$ (refer to (40)).

Noticing $\left|\sum_{j\in D_N} b_j\right| \leq 1/C^*L_N$ and using (20), we have

$$|T_1 \prod_{k=1}^{N} \rho_k| \leq C|\sum_{k=1}^{N} b_k^3 \hat{w}_k \prod_{m=1,\, m\neq k} \rho_m| \leq C\, L_N^2 \exp\{-\mu(\psi^2+t^2)\},$$

$$|T_2 \prod_{k=1}^{N} \rho_k| \leq C \sum_{k=1}^{N} b_k^2 |\hat{v}_k| |\sum_{j\neq k,\, j\in H_N} b_j \hat{u}_j| \prod_{m\neq k} |\rho_m| +$$

$$+ C \sum_{k=1}^{N} b_k^2 |\hat{v}_k| \sum_{j\in D_N,\, j\neq k} |b_j \hat{u}_j| \prod_{m=1,m\neq k,j}^{N} |\rho_m|$$

$$\leq C \sum_{k=1}^{N} b_k^2 \sum_{j\neq k,\, j\in H_N} |b_j \xi_j| \prod_{m\neq k} |\rho_m| + C \sum_k b_k^2 \sum_{j\in D_N} |b_j| \prod_{\substack{m=1 \\ m\neq k,j}}^{N} |\rho_m|$$

$$\leq C\, L_N^3 (C^{*-1} L_N^{-1} + |\psi| + |t|) \exp\{-\mu(\psi^2+t^2)\}$$

$$\leq C\, L_N^2 (1+|t|+|\psi|) \exp\{-\mu(\psi^2+t^2)\}.$$

By the similar method the following estimate also can be inferred:

$$|T_3 \prod_{k=1}^{N} \rho_k(\psi,t)| \leq C\{L_N^2 Q_2(|\psi|) + L_N Q_3(|\psi|,|t|)\} \exp\{-\mu(\psi^2+t^2)\}.$$

Note that above estimate is also valid for $(i^3 E\, S_n^3 + 3t - t^3)\exp\{-\tfrac{1}{2}(\psi^2+t^2)\}$ when $|t| \geq C''\sqrt{pq}\, b_*^{-1}$, so (47) holds for $(\psi,t)\in\Gamma_3$. Hence, when $|t| \leq C''\sqrt{pq}\, L_N^{-1}$, we have

$$\left| \frac{1}{\sqrt{2\pi}} \int_{|\psi|<\pi\sqrt{Npq}} [(T_1+T_2+T_3) \prod_{k=1} \rho_k(\psi,t) - (i^3 E S_n^3 + 3t - t^3) e^{-\frac{1}{2}(\psi^2+t^2)}] d\psi \right|$$

$$\leq C(L_N^2 + L_N |t| Q_1(|t|)) \exp\{-\mu t^2\}.$$

By (36) and the equality $\sqrt{Npq}\, B_n(p) = 1+O(N^{-1})$, it holds that

$$|i^3E(S_n^3e^{itS_n}) - \psi_n(t)| \le C(L_N^2+L_N|t|Q_1(|t|)\exp\{-\mu t^2\} \tag{48}$$

for $|t| \le C''\sqrt{pq}\, L_N^{-1}$.

When $|t| \le C\, L_N$, from lemma 1, we get

$$|t|^{-1}|i^3E(S_n^3e^{itS_n}) - (i^3E\ S_n^3+3t-t^3)e^{-\frac{t^2}{2}}|$$

$$\le |t|^{-1}\{|ES_n^3(e^{itS_n}-1)| + |ES_n^3(1-e^{-t^2/2})| + (3|t|+|t|^3)e^{-t^2/2}\}$$

$$\le E\ S_n^4 + \tfrac{1}{2}|t|E|S_n^3| + 3|t| + |t|^3 \le Q_1(|t|). \tag{49}$$

With (48) and (49) we obtain

$$\int_{|t|\le C''\sqrt{pq}L_N^{-1}} |t|^{-1}|i^3ES_n^3e^{itS_n}-\psi_n(t)|dt = \{\int_{|t|\le CL_N} +$$

$$+ \int_{CL_N\le|t|\le C''\sqrt{pq}L_N^{-1}}\}|t|^{-1}|i^3E\ S_n^3e^{itS_n} - \psi_n(t)|dt$$

$$\le C\ L_N.$$

Up to now the lemma is proved.

Lemma 8. Let $I = n-J = [\sqrt{n}]$. Under the condition of the theorem 1, the following relation is valid:

$$E\{(S_n+\Delta_{n1}^*)^3\exp\{it(S_n+\Delta_{n1}^*)\}\sim E\{(S_n+\Delta_{n1}^*)^3\exp(itS_n)\}.$$

Proof. We only prove the relations

$$E\{S_n^{3-m}\Delta_{n1}^{*m}\ \ e^{itS_n}(e^{it\Delta_{n1}^*}-1)\}\sim 0,\quad \text{for } m = 0,1,2,3. \tag{50}$$

From Jensen's inequality, for $\alpha \geq 1$, we have $E|g_1^*|^\alpha = E(|E(\phi_{12}^*|X_1)|^\alpha)$

$$\leq E\{E(|\phi_{12}^*|^\alpha|X_1)\} = E|\phi_{12}^*|^\alpha \leq CE|\hat{\phi}_{12}|^\alpha.$$

so with lemma 2, we get

$$E\Delta_{n1}^{*4} \leq Cn^{-3/2}\nu_3.$$

Using lemma 1 and 2 and Hölder's inequality, we have

$$E|S_n^{3-m}\Delta_{n1}^{*\,m+1}| \leq Cn^{(m+1)/2}\nu_3, \text{ for } m = 1,2.$$

Hence

$$|t|^{-1}|E\ S_n^{3-m}\Delta_{n1}^{*\,m}e^{itS_n}(e^{it\Delta_{n1}^*}-1)| \leq E|S_n^{3-m}\Delta_{n1}^{*\,m+1}| \leq CN^{-1}\nu_3$$

for $m = 1,2,3$. From this (50) holds for $m = 1,2,3$.

Now we prove the case of $m = 0$. It is obvious that there exist θ_j, $|\theta_j| \leq 1, j = 1,2$ such that

$$e^{it\Delta_{n1}^*}-1 = it\Delta_{n1}^* + 2\theta_1|t\Delta_{n1}^*|I(A_J \cup B_J) + \theta_2 t^2\Delta_{n1}^{*2}I(A_J^c \cap B_J^c),$$

here the definition of A_J and B_J can be found in (11) and (12). So we have

$$|t|^{-1}|E(S_n^3 e^{itS_n}(e^{it\Delta_{n1}^*}-1))| \leq |E(S_n^3 e^{itS_n}\Delta_{n1}^*)| +$$

$$+ 2E(|S_n^3\Delta_{n1}^*|I(A_J\cup B_J)) + |t|\left|E(\theta_2\Delta_{n1}^{*2}S_n^3 e^{itS_n} I(A_J^c\cap B_J^c))\right|$$

$$\triangleq \sum_{j=1}^{3} M_j(t). \tag{51}$$

Using lemma 1, 3, 4 and Hölder's inequality and noticing $\nu_3 \geq \sigma_g^3 = 1$, we obtain

$$M_2(t) \leq 2(E|S_n|^{18})^{1/6}(E|\Delta_{n1}^*|^3)^{1/3}[P(A_J\cup B_J)]^{\frac{1}{2}}$$

$$\leq CN^{-\frac{1}{2}}\nu_3 L_N. \tag{52}$$

So

$$\int_{|t|\leq\lambda L_N^{-1}} M_2(t)dt \leq C\, N^{-\frac{1}{2}}\nu_3.$$

From lemma 1, 3, 6 and Hölder's inequality, for appropriate selected ε and ε^*, there exists λ such that

$$\int_{|t|\leq\lambda L_N^{-1}} M_3(t)dt \leq C\int_{|t|\leq\lambda L_N^{-1}} |t| \sum_{r=0}^{3} E\{\Delta_{n1}^{*2}|S_n|^{3-r}I(A_J^c\cap B_J^c)\,\cdot$$

$$|E[(S_n'')^r e^{itS_n''}|X_1,\ldots,X_J]|\}dt$$

$$\leq C\int_{|t|\leq\lambda L_N^{-1}} |t| \sum_{r=0}^{3} E\,\Delta_{n1}^{*2}|S_n'|^{3-r}(1+|S_n'|^3+N^{-\frac{3}{4}}|t|^3)\exp\{-\mu N^{-\frac{1}{2}}t^2\}dt$$

$$\leq C\int_{|t|\leq\lambda L_N^{-1}} N^{-1}|t|\nu_3(1+N^{-\frac{3}{4}}|t|^3)\exp\{-\mu N^{-\frac{1}{2}}t^2\}dt$$

$$\leq C\int_0^\infty |\upsilon| N^{-\frac{1}{2}}\nu_3(1+|\upsilon|^3)\exp\{-\mu\upsilon^2\}d\upsilon \leq CN^{-\frac{1}{2}}\nu_3. \tag{53}$$

Set

$$\tilde{J} = [n/2],\ \lambda_n = \binom{J}{2}/\binom{\tilde{J}}{2},\ \tilde{S}_n' = \sum_{j=1}^{\tilde{J}}\xi_j,\ \tilde{S}_n'' = \sum_{j=\tilde{J}+1}^{N}\xi_j,$$

$$\tilde{\Delta}_{n1}^* = d_n \sum_{1\leq j<k\leq\tilde{J}} Y_{jk}^*.$$

By the symmetry, we have

$$M_1(t) = \lambda_n|E(\tilde{\Delta}_{n1}^* S_n^3 e^{itS_n})| \leq C\ E\{|\tilde{\Delta}_{n1}^* S_n^3| I(A_{\tilde{J}} \cup B_{\tilde{J}})\} +$$

$$+ C\sum_{r=0}^{3} E\{|\tilde{\Delta}_{n1}^*(\tilde{S}_n')^{3-r}| I(A_{\tilde{J}}^c \cap B_{\tilde{J}}^c)|E[(\tilde{S}_n'')^r e^{it\tilde{S}_n''}|X_1,\ldots,X_J]|\}$$

$$\triangleq M_{11}(t) + M_{12}(t), \tag{54}$$

Where $A_{\tilde{J}}$, $B_{\tilde{J}}$ were defined by (11) and (12). Using an argument similar to that employed in establishing (52) and (53), we see that for appropriate selected ε and ε^*, there exists λ such that

$$\int_{|t|\leq\lambda L_N^{-1}} M_{11}(t)dt \leq CN^{-\frac{1}{2}}\nu_3, \tag{55}$$

$$\int_{|t|\leq\lambda L_N^{-1}} M_{12}(t)dt \leq C\int_{|t|\leq\lambda L_N^{-1}} \sum_{r=0}^{3} E\{|\tilde{\Delta}_{n1}^*(\tilde{S}_n')^{3-r}\ (1+|\tilde{S}_n'|^3+|t|^3)\}e^{-\mu t^2}dt$$

$$\leq CN^{-\frac{1}{2}}\nu_3\int_{|t|\leq\lambda L_N^{-1}}(1+|t|^3)e^{-\mu t^2}dt \leq CN^{-\frac{1}{2}}\nu_3. \tag{56}$$

From (51) to (56), (50) is proved for m = 0, thus the lemma 8 holds.

Lemma 9. Let $I = n - J = [\sqrt{n}]$. Under the condition of the theorem 1, we have

$$i^3 E\{(S_n+\Delta_{n1}^*)^3 e^{itS_n}\} \sim \beta_n(t) = \{i^3 E(S_n+\Delta_{n1}^*)^3 + 3t - t^3\} e^{-\frac{t^2}{2}} .$$

Proof. From lemma 7, we need only to prove

$$E(S_n^{3-m}\Delta_{n1}^* e^{itS_n}) \sim E(S_n^{3-m}\Delta_{n1}^{*\,m}) e^{-t^2/2}, \text{ for } m = 1,2,3 \tag{57}$$

But from lemma 1, 3 and Hölder's inequality, it is obtained that

$$|t|^{-1} \left| E\, S_n^{3-m}\Delta_{n1}^{*m} (e^{itS_n} - e^{-t^2/2}) \right|$$

$$\leq |t|^{-1} E|S_n^{3-m}\Delta_{n1}^{*\,m}(e^{itS_n}-1)| + |t|^{-1} E|S_n^{3-m}\Delta_{n1}^{*\,m}(1-e^{-t^2/2})|$$

$$\leq E|S_n^{4-m}\Delta_{n1}^{*\,m}| + E|S_n^{3-m}\Delta_{n1}^{*\,m}| \leq C\, N^{-\frac{m}{2}} \nu_3,$$

for $m = 2, 3$. From this estimate we see that (57) holds for $m = 2, 3$. In order to prove the case of $m = 1$, taking $\tilde{J} = [\frac{n}{2}]$ and introducing $\tilde{S}_n'$, $\tilde{S}_n''$ and $\tilde{\Delta}_{n1}^*$, as the proof of the lemma 8, we need only to prove

$$E(S_n^2\tilde{\Delta}_{n1}^* e^{itS_n}) \sim E(S_n^2\tilde{\Delta}_{n1}^*) e^{-\frac{t^2}{2}} . \tag{58}$$

Using the similar method employed in establishing (54)-(56), we know that there exists λ such that

$$\int_{1\le|t|\le\lambda L_N^{-1}}|t|^{-1}|E(S_n^2\tilde{\Delta}_{n1}^*)e^{itS_n}|dt \le \int_{1\le|t|\le\lambda L_N^{-1}}|ES_n^2\tilde{\Delta}_{n1}^* e^{itS_n}|dt$$

$$\le \int_{1\le|t|\le\lambda L_N^{-1}}\{E|S_n^2\tilde{\Delta}_{n1}^* I(A_{\tilde{j}}\cup B_{\tilde{j}})| + |ES_n^2\tilde{\Delta}_{n1}^* e^{itS_n} I(A_{\tilde{j}}^c\cap B_{\tilde{j}}^c)|\}dt$$

$$\le c\, N^{-\frac{1}{2}}\nu_3. \tag{59}$$

But

$$\int_{1\le|t|\le\lambda L_N^{-1}}|t|^{-1}|E(S_n^2\tilde{\Delta}_{n1}^*)e^{-\frac{t^2}{2}}|dt \le C\, N^{-\frac{1}{2}}\nu_3, \tag{60}$$

$$\int_{|t|\le1}|t|^{-1}|E\, S_n^2\tilde{\Delta}_{n1}^*(e^{itS_n}-e^{-t^2/2})|dt$$

$$\le \int_{|t|\le1}|t|^{-1}(|ES_n^2\tilde{\Delta}_{n1}^*(e^{itS_n}-1)| + |ES_n^2\tilde{\Delta}_{n1}^*(1-e^{-t^2/2})|)dt$$

$$\le \int_{|t|\le1}(E|S_n^3\tilde{\Delta}_{n1}^*|+E|S_n^2\tilde{\Delta}_{n1}^*|)dt \le C\, N^{-\frac{1}{2}}\nu_3, \tag{61}$$

so the relation (58) holds by (59)-(61), and the lemma 9 is proved.

<u>Lemma 10</u>. Suppose that $\psi(t)$ have continuous third-order derivative $\psi^{(3)}(t)$ in $|t| \le T$, and $\psi^{(j)}(0) = 0$ for $j = 0, 1, 2,$. Then

$$\int_{-T}^{T} |t|^{j-4}|\psi^{(j)}(t)|dt \le \int_{-T}^{T}|t|^{-1}|\psi^{(3)}(t)|dt, \text{ for } j = 0, 1, 2.$$

The proof can be referred to lemma 2 of the paper[3].

<u>Lemma 11.</u> Suppose that (2) is valid. Let $\{W_{n1}\}$ and $\{W_{n2}\}$, $n = 1, 2, \ldots$, be a sequences of random variables, $W_n = W_{n1} + W_{n2}$, and $\{a_n\}$ be a sequence of real numbers such that $|a_n-1| \le C/\sqrt{N}$. Then, the following conclusions are valid.

(1). If we have

$$|P(W_{n1} \leq x)-\Phi(x)| \leq C\, N^{-\frac{1}{2}}\nu_3(1+|x|)^{-3} \qquad (62)$$

for all x and n, and

$$P(|W_{n2}| \geq C|x|\ /\sqrt{N}) \leq C\, N^{-\frac{1}{2}}\nu_3(1+|x|)^{-3},$$

for all $|x| \geq 1$. Then

$$|P(W_n \leq x) - \Phi(x)| \leq CN^{-\frac{1}{2}}\nu_3(1+|x|)^{-3}, \text{ for all } x \text{ and } n. \qquad (63)$$

(2). Suppose that $\nu_3 \geq C\sqrt{N}$, also (62) and the following hold,

$$P(|W_{n2}| \geq \tfrac{1}{2}|x|) \leq C\, N^{-\frac{1}{2}}\nu_3|x|^{-3}$$

for $|x| \geq 1$. Then (63) holds.

Proof. Refer to the proof of lemma 1 in the paper[3], and use the condition "$1 \leq C^{-1}N^{-\frac{1}{2}}\nu_3$".

III. Proof of the Theorem

In order to prove the theorem 1, first we prove the following theorem:

Theorem 2. Let $\sum_{j=1}^{N} b_j = 0$, $\sum_{j=1}^{N} b_j^2 = 1$ and (2) hold. Then for all n and x we have

$$|P(S_n \leq x) - \Phi(x)| \leq C\, L_N(1+|x|)^{-3}.$$

Proof. Set $\alpha_0 = 1$, $\alpha_1 = 0$, $\alpha_2 = ES_n^2 - 1$, $\alpha_3 = ES_n^3$. Obviously, $|\alpha_j| \leq CL_N$ are valid for j = 1, 2, 3. Define

$$h_n(t) = \sum_{k=0}^{3} \alpha_k (it)^k e^{-t^2/2}/k!, \tag{64}$$

$$g_n(t) = E\, e^{itS_n}. \tag{65}$$

Obviously,

$$|h_n(t) - e^{-t^2/2}| \leq CL_N(t^2+|t|^3)e^{-t^2/2},$$

and

$$|h_n^{(3)}(t) - \psi_n(t)| \leq CL_N(|t|+t^6)e^{-t^2/2},$$

where the definition of $\psi_n(t)$ is found in the lemma 7. With the lemma 3 of [5] and the lemma 7 there exists $\lambda > 0$ such that

$$\int_{|t|\leq\lambda L_N^{-1}} |t|^{-1}|g_n(t) - h_n(t)|dt \leq C\, L_N, \tag{66}$$

$$\int_{|t|\leq\lambda L_N^{-1}} |t|^{-1}|g_n^{(3)}(t) - h_n^{(3)}(t)|dt \leq C\, L_N. \tag{67}$$

Noticing $g_n^{(j)}(0) = h_n^{(j)}(0)$ for $j = 0, 1, 2$, with lemma 10 and (67), we get

$$\int_{|t|\leq\lambda L_N^{-1}} |t|^{j-4}|g_n^{(j)}(t) - h_n^{(j)}(t)|dt \leq C\, L_N, \text{ for } j = 0,1,2,3. \tag{68}$$

Define

$$G_n(x) = P(S_n \leq x),\ H_n(x) = \sum_{k=0}^{3} \frac{(-1)^k}{k!} \alpha_k \Phi^{(k)}(x). \tag{69}$$

It is easy to see that $G_n(x)$ is non-decreasing, $H_n(x)$ is differential and has bounded variation on R^1, and $G_n(\pm\infty) = H_n(\pm\infty)$, $\int_{-\infty}^{\infty} |x|^3 |d(G_n(x)-H_n(x))| < \infty$, $|H_n'(x)| \leq C(1+|x|)^{-3}$. So G_n and H_n satisfy the conditions of lemma 8 of the Chapter 6 in [2]. Checking the proof of the lemma again, we see that this lemma also holds when $T \geq \lambda$, here λ is an absolute constant.

Set

$$\delta_3(t) = \int_{-\infty}^{\infty} e^{itx} d(x^3(G_n(x)-H_n(x)),$$

then

$$|G_n(x) - H_n(x)| \leq C(1+|x|)^{-3}\{\int_{|t|\leq\lambda L_N^{-1}} |t|^{-1}|g_n(t) - h_n(t)|dt + \int_{|t|\leq\lambda L_N^{-1}} |t|^{-1}|\delta_3(t)|dt + C L_N\}. \tag{70}$$

From the lemma 7 of the Chapter 6 in [2], and noticing (66), (68) and (70), we get

$$\begin{aligned}|G_n(x) - H_n(x)| &\leq C(1+|x|)^{-3}\{\int_{|t|\leq\lambda L_N^{-1}} |t|^{-1}|g_n(t) - h_n(t)|dt \\ &+ \sum_{j=0}^{3} \int_{|t|\leq\lambda L_N^{-1}} |t|^{j-4}|g_n^{(j)}(t) - h_n^{(j)}(t)|dt + C L_N\} \\ &\leq C L_N(1+|x|)^{-3}.\end{aligned} \tag{71}$$

But

$$|H_n(x) - \Phi(x)| \leq C L_N(1+|x|)^{-3},$$

so the theorem 2 is obtained from (71).

In the following we give the proof of the theorem 1.

Proof. First suppose that $\nu_3 \geq \sqrt{N}$. By lemma 3, we have

$$P(|\Delta_n| \geq \tfrac{1}{2}|x|) \leq C|x|^{-3}E|\Delta_n|^3 \leq C|x|^{-3}.N^{-3/2}\nu_3,$$

for $|x| \geq 1$. Using lemma 11 (2) and theorem 1, and noticing $\tilde{U}_n = S_n + \Delta_n$, we get

$$|P(\tilde{U}_n \leq x) - \Phi(x)| \leq C\, N^{-\frac{1}{2}}\nu_3(1+|x|)^{-3}. \tag{72}$$

Now we suppose that $\nu_3 < \sqrt{N}$, write

$$\alpha_0 = 1,\ \alpha_1 = 0,\ \alpha_2 = E(S_n+\Delta^*_{n1})^2 - 1,\ \alpha_3 = E(S_n+\Delta^*_{n1})^3,$$

$$h_n(t) = \sum_{k=0}^{3} \alpha_k (it)^k e^{-t^2/2}/k!, \tag{73}$$

$$g_n(t) = E\,\exp\{it(S_n+\Delta^*_{n1})\}, \tag{74}$$

and

$$\beta_n(t) = \{i^3E(S_n+\Delta^*_{n1})^3 + 3t - t^3\}e^{-t^2/2}.$$

Clearly, $h_n(t) \sim e^{-t^2}$, and by lemma 9, we have $h_n^{(3)}(t) \sim \beta_n(t)$. From the proof of the theorem 1 in [5], we have $g_n(t) \sim e^{-t^2/2}$. Using lemma 8 and lemma 9, we get $g_n^{(3)}(t) \sim \beta_n(t)$. Hence $g_n(t) \sim h_n(t)$, $g_n^{(3)}(t) \sim h_n^{(3)}(t)$ and $g_n^{(j)}(0) = h_n^{(j)}(0)$ for $j = 0, 1, 2$. Similar to the proof of the theorem, we get that there exists $\lambda > 0$ such that

$$\int_{|t|\le\lambda L_N^{-1}} |t|^{-1}|g_n(t) - h_n(t)|dt \le C\, N^{-\frac{1}{2}}\nu_3,$$

$$\int_{|t|\le\lambda L_N^{-1}} |t|^{j-4}|g_n^{(j)}(t) - h_n^{(j)}(t)|dt \le C\, N^{-\frac{1}{2}}\nu_3, \text{ for } j=0,1,2,3.$$

Similar to the proof of the theorem 2, we can obtain

$$|P(S_n+\Delta_{n1}^* \le x) - \Phi(x)| \le C\, N^{-\frac{1}{2}}\nu_3(1+|x|)^{-3}. \tag{75}$$

When $|x| \ge 1$, with (9) and (10) we get

$$P(|\Delta_{n2}^*| \ge c|x|/\sqrt{N}) \le C\, N^{3/2}|x|^{-3}E|\Delta_{n2}^*|^3$$

$$\le C\, N^{3/2}|x|^{-3}.CN^{-9/2}(\sqrt{n}\cdot n)^{3/2}\nu_3 \le C\, N^{-\frac{1}{2}}\nu_3|x|^{-3}.$$

Hence with (1) of lemma 11 and (75), we have

$$|P(S_n+\Delta_n^* \le x) - \Phi(x)| \le C\, N^{-\frac{1}{2}}\nu_3\,(1+|x|)^{-3}. \tag{76}$$

Set

$$\tilde{\phi}_{jk} = \phi_{jk}I(|\phi_{jk}|\le\sqrt{n}(1+|x|)),\ \tilde{\phi}_{jk}^* = \tilde{\phi}_{jk} - E\tilde{\phi}_{jk},\ j \ne k,\ \tilde{g}_j^* = E(\tilde{\phi}_{jk}^*|X_j),$$

$$\tilde{Y}_{jk}^* = \tilde{\phi}_{jk}^* - \frac{N-1}{N-2}(\tilde{g}_j^*+\tilde{g}_k^*),\ \tilde{\Delta}_n^* = d_n \sum_{1\le j<k\le n}\tilde{Y}_{jk}^*,\ Z_{jk} = \tilde{Y}_{jk}^* - Y_{jk}^*,$$

then

$$E\, Z_{12}^4 \le C\sqrt{n}(1+|x|)\nu_3.$$

Using Jensen's inequality, we get

$$E\, Z_{12}^2 \le 3\{E(\tilde{\phi}_{12}^{*}-\phi_{12}^{*})^2 + 2E(E[(\tilde{\phi}_{12}^{*}-\phi_{12}^{*})|X_1])^2\}$$

$$\le 3\{E(\tilde{\phi}_{12}^{*}-\phi_{12}^{*})^2 + 2E(E[(\tilde{\phi}_{12}^{*}-\phi_{12}^{*})^2|X_1])\}$$

$$\le 9\, E(\tilde{\phi}_{12}^{*}-\phi_{12}^{*})^2$$

$$\le 9\, E\phi_{12}^2 I(\sqrt{n}<|\phi_{12}|\le\sqrt{n}(1+|x|)) \le 9n^{-\frac{1}{2}}E|\phi_{12}|^3.$$

Hence, from lemma 2 and the supposition $\nu_3 < \sqrt{n}$, i.e. $n^{-\frac{1}{2}}\nu_3 \le C$, we have

$$E(\tilde{\Delta}_n^{*}-\Delta_n^{*})^4 \le C\, n^{-6}E(\sum_{1\le j<k\le n} Z_{jk})^4 \le Cn^{-3}EZ_{12}^4 + Cn^{-2}(EZ_{12}^2)^2$$

$$\le Cn^{-3}\cdot c\sqrt{n}(1+|x|)\nu_3 + Cn^{-2}(9n^{-\frac{1}{2}}\nu_3)^2$$

$$\le Cn^{-5/2}\nu_3(1+|x|). \tag{77}$$

On the other hand, if we set $W_{jk} = Y_{jk} - \tilde{Y}_{jk}^{*}$, then it is easy to see that

$$E(\Delta_n-\tilde{\Delta}_n^{*})^2 \le c\, n^{-1}EW_{12}^2 \le C\, n^{-1}\cdot 9E\phi_{12}^2 I(|\phi_{12}|>\sqrt{n}(1+|x|))$$

$$\le C\, n^{-3/2}\nu_3(1+|x|)^{-1}. \tag{78}$$

Thus, from (77) and (78), we have

$$P\{|\tilde{U}_n-(S_n+\Delta_n^*)| \geq |x|/\sqrt{n}\}$$

$$\leq P(|\Delta_n-\tilde{\Delta}_n^*| \geq |x|/2\sqrt{n}) + P(|\tilde{\Delta}_n^*-\Delta_n^*| \geq |x|/2\sqrt{n}\}$$

$$\leq 4nx^{-2}E(\Delta_n-\tilde{\Delta}_n^*)^2 + 16\, n^2x^{-4}E(\tilde{\Delta}_n^*-\Delta_n^*)^4$$

$$\leq C\, n^{-\frac{1}{2}}\nu_3(1+|x|)^{-3}, \tag{79}$$

for all $|x| \geq 1$. Further, with (1) of lemma (11) and (76) and (79), we get

$$|P(\tilde{U}_n \leq x) - \Phi(x)| \leq C\, n^{-\frac{1}{2}}\nu_3(1+|x|)^{-3}. \tag{80}$$

Noticing $\tilde{U}_n = \frac{N-2}{N-1}(\frac{\sqrt{n}U_n}{2\sqrt{q}})$, and using lemma 11 (1), we get

$$|P(\frac{\sqrt{n}U_n}{2\sqrt{q}} \leq x) - \Phi(x)| \leq C\, n^{-\frac{1}{2}}\nu_3\, (1+|x|)^{-3}, \text{ for all } x \text{ and } n,$$

i.e. (3) holds for $\sigma_g = 1$. So the theorem 1 is also valid in general case.

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